Celestial CFT Correlators and Conformal Block Decomposition

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based on:

W. Fan, A.F., T.R. Taylor

Soft Limits of Yang-Mills Amplitudes and Conformal Correlators

arXiv:1903.01676, JHEP 05 (2019) 121

A.F. S.Stieberger., T.R. Taylor, Bin Zhu:

BMS Algebra from Soft and Collinear Limits

arXiv:1912.10973, JHEP 03 (2020) 130

Extended Super BMS Algebra of Celestial CFT

arXiv:2007.03785 JHEP 09 (2020) 198

Wei Fan, A.F., S. Stieberger T.R. Taylor:

On Sugawara construction on Celestial Sphere

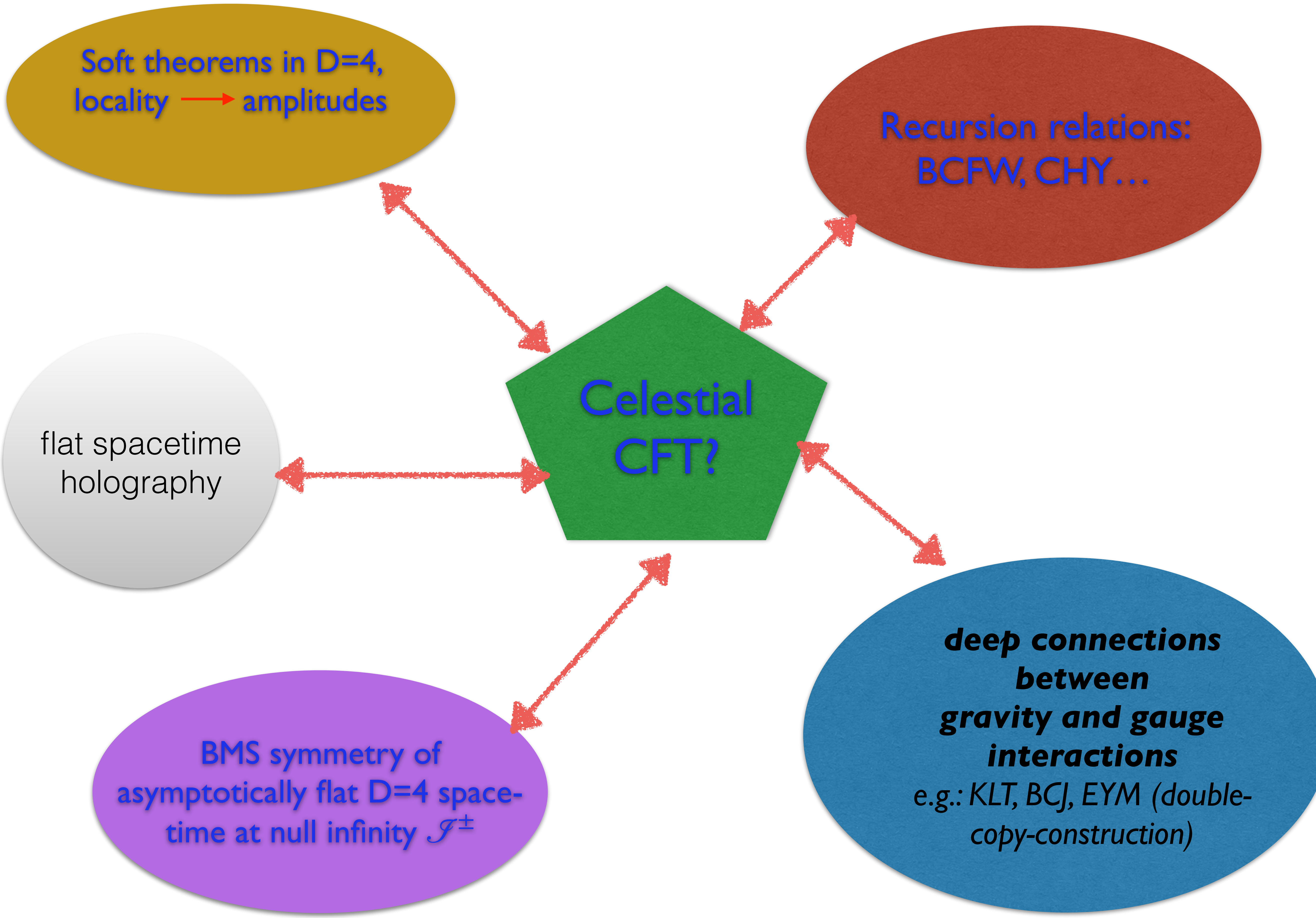
arXiv:2005.10666 JHEP 09 (2020) 139

Wei Fan, A.F., S. Stieberger, T.R. Taylor, Bin Zhu:

Conformal Blocks from Celestial Gluon Amplitudes

arXiv:2103.0442

Why study CCFT ?



$$ds^2 = -\,dt^2 + d\vec{x}^{\,2}$$

Flat Minkowski metric in retarded (or Bondi) coordinates $(u, r, z, \bar{z})$

$$\begin{cases} x^0 = u + r \\ x^1 = \dfrac{r\,(z+\bar{z})}{1+|z|^2} \\ x^2 = -\,i\dfrac{r\,(z-\bar{z})}{1+|z|^2} \\ x^3 = \dfrac{r(1-|z|^2)}{1+|z|^2} \end{cases} \qquad r^2 = \vec{x}^{\,2}$$

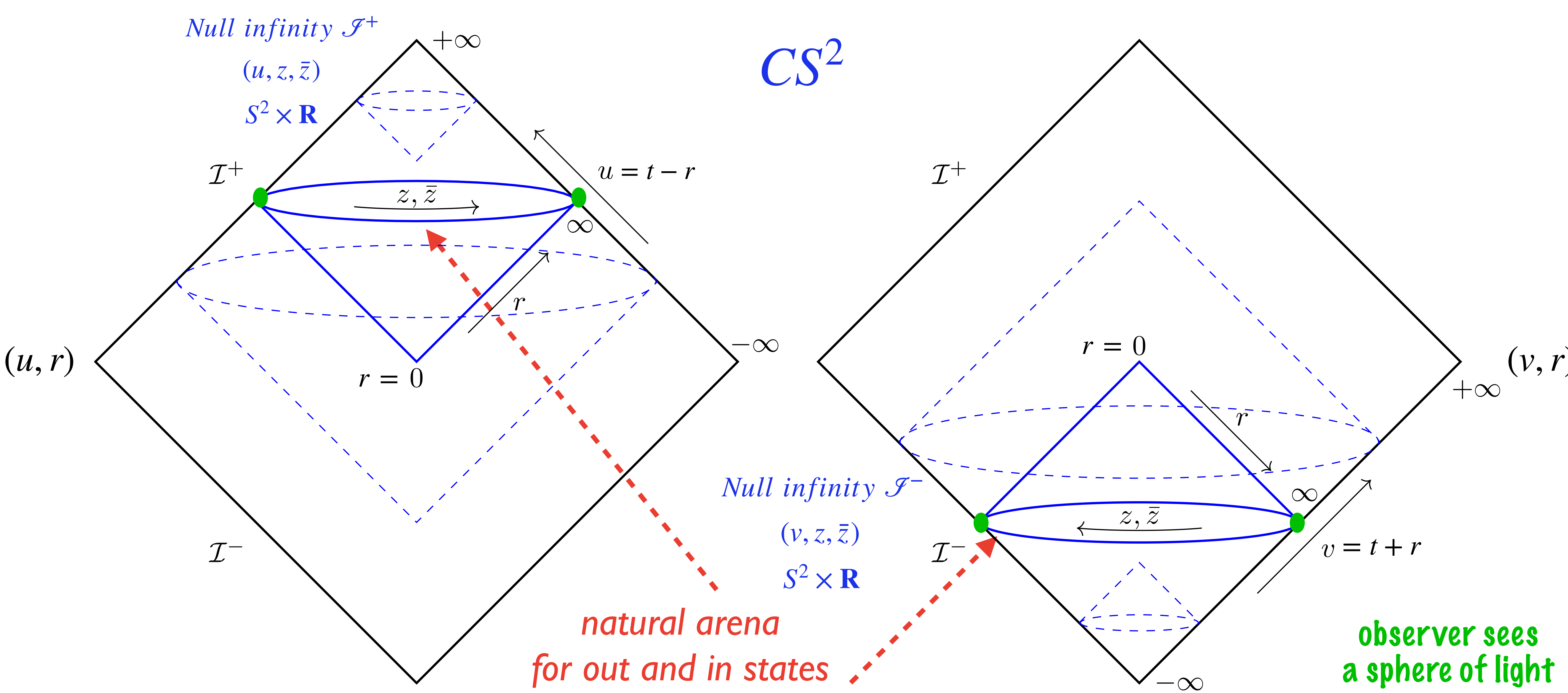
Null infinity $\mathscr{I}^+$

$S^2 \times \mathbf{R}$

$(u, z, \bar{z})$

$$ds^2 = -\,du^2 - 2\,du dr + \underbrace{\frac{4r^2}{(1+|z|^2)^2}}\,dz d\bar{z}$$

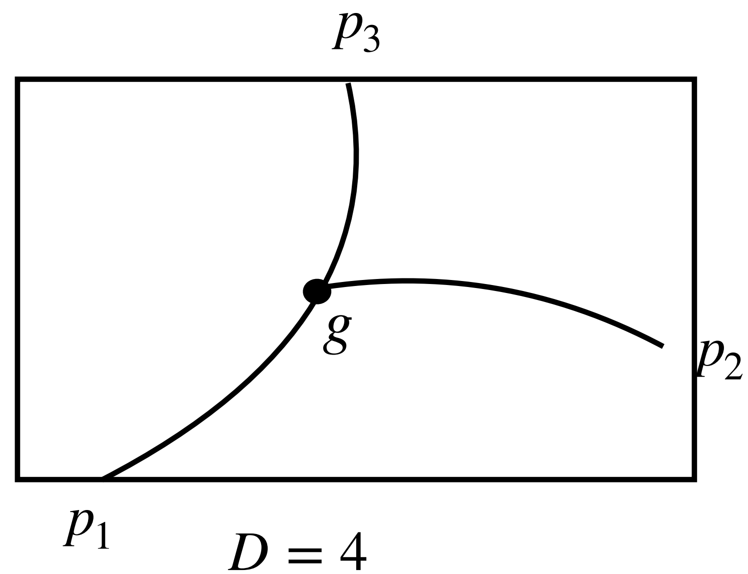
CS^2



Basic Idea

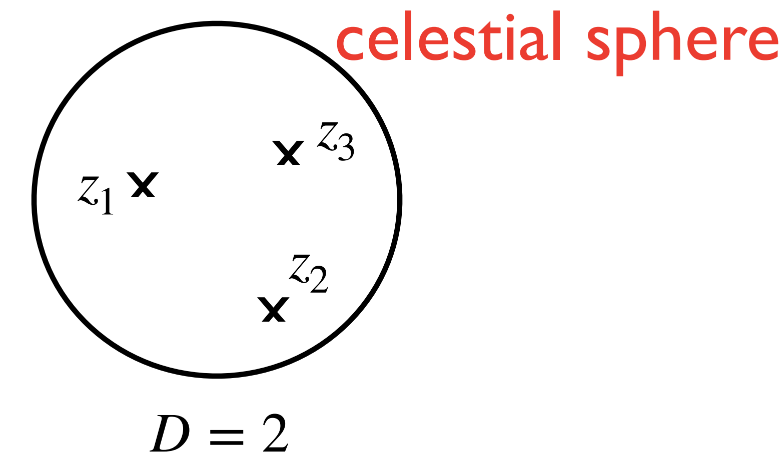
Amplitudes = conformal correlators of primary fields on celestial sphere

traditional amplitudes describe <u>transitions</u> between <u>momentum</u> <u>eigenstates</u>



$$z_k = \frac{p_k^1 + i p_k^2}{p_k^0 + p_k^3}$$

=



D=2 Euclidean CFT

D=4 space-time QFT

$$\mathscr{A}(\{p_i, \epsilon_j\}) = i(2\pi)^4\ \delta^{(4)}\left(p_1 + p_2 + p_3\right)\ A(\{p_i, \epsilon_j\})$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle \sim \frac{g}{|z_1 - z_2|^{h_1+h_2-h_3}\ |z_2 - z_3|^{h_2+h_3-h_1}\ |z_1 - z_3|^{h_1+h_3-h_2}}$$

Lorentz symmetry

$$SO(1,3) \simeq SL(2,\mathbf{C})$$

$$z_i \to \frac{a z_i + b}{c z_i + d}$$

global conformal symmetry on CS^2

We will see later that the global conformal groups gets enhanced to the full local conformal group $z \to w(z)$

CCFT: Massless particles on celestial sphere

$$p^\mu = \omega \; q^\mu(z, \bar{z}) \qquad q^\mu = (1 + |z|^2, z + \bar{z}, -i(z - \bar{z}), 1 - |z|^2)$$

In the massless case, transition from momentum space to conformal primary wavefunctions (CPW) with conformal dimension Δ is implemented by Mellin transform:

$$|\Delta, z\rangle = \int_0^\infty d\omega \; \omega^{\Delta-1} \; |\omega, z\rangle$$

Pasterski, Shao (2017),

also Banerjee (2018)

$$\tilde{\phi}(\Delta, z, \bar{z}; x,) = \int_0^\infty d\omega \; \omega^{\Delta-1} \phi(\omega, z, \bar{z}; x) \qquad \Delta = 1 + i\lambda, \; \lambda \in \mathbf{R}$$

described by
- the point $z \in CS^2$ at which it enters or exits the celestial sphere
- SL(2,C) Lorentz quantum numbers $(h, \bar{h})$

CCFT: Massless particles on celestial sphere

E.g.: scalar plane wave $e^{\pm ip\cdot x}$

$$\varphi^{\pm}_{\Delta}(x,z,\bar{z}) = \int_0^\infty d\omega\ \omega^{\Delta-1}\ e^{\pm i\omega q_\mu x^\mu - \epsilon\omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(q(z,\bar{z})\cdot x \mp i\epsilon)^\Delta}$$

solves D=4 Klein-Gordon equation

Bases	Plane waves	Conformal Primary Wave Functions
Notation	$exp(\pm ip\cdot X)$	$\varphi^{\pm}_{\Delta}(X,z,\bar{z})$
Labels	$p^\mu,\ (p^2=0,\ p^0>0)$	$\Delta = 1+i\lambda, (\lambda\in\mathbb{R})$ $z\in CS^2$

$$\varphi_\Delta\left(\Lambda^\mu{}_\nu x^\nu, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{c}\bar{z}}{\bar{c}\bar{z}+\bar{d}}\right) = \left(cz+d\right)^{\Delta/2}\left(\bar{c}\bar{z}+\bar{d}\right)^{\Delta/2}\varphi_\Delta(x^\mu,z,\bar{z})$$

gauge boson: $\epsilon_\mu e^{ip\cdot x}$

$$V^{\Delta,\pm}_{\mu,J}(x^\mu, z, \bar{z}) \equiv \frac{\partial_\ell q_\mu}{\sqrt{2}} \int_0^\infty d\omega\ \omega^{\Delta-1} e^{\pm i\omega q\cdot x - \epsilon\omega} \qquad (\ell = z, \bar{z}; J = \pm 1),$$

solves Maxwell in D=4

Bases	**Plane waves**	**Conformal Primary Wave Functions**
Notation	$A_{\mu\ell}(x,p) = \epsilon_{\mu\ell}(p) exp(\pm i p_\mu x^\mu)$	$V^{\Delta\pm}_{\mu J} = (\pm i)^\Delta \frac{\Gamma(\Delta)}{\sqrt{(2)}} \frac{\partial_\ell q^\mu}{(-q_\mu x^\mu \mp \epsilon)^\Delta}$
3 continuous parameters	$p^\mu,\ (p^2 = 0,\ p^0 > 0)$	$\Delta = 1 + i\lambda, (\lambda \in \mathbb{R})$ $z \in CS^2$
2 discrete parameters	4d helicity $\ell = \pm 1$ incoming vs outgoing	2d spin $J = \pm 1$ incoming vs outgoing

$$\partial_\ell q^\mu(z, \bar{z}) = \begin{cases} \partial_z q^\mu = \sqrt{(2)}\epsilon^\mu_+(q) = (z, 1, -i, \bar{z}) \\ \partial_{\bar{z}} q^\mu = \sqrt{(2)}\epsilon^\mu_-(q) = (z, 1, +i, \bar{z}) \end{cases}$$

Particles <-> Operators

in momentum basis: plane waves with momentum $p = \omega q(z)$

in conformal basis: conformal primary wave functions $\Phi \to \mathcal{O}$

"state operator correspondence"

$$\mathcal{O}_{h,\bar{h}}\left(\frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = (cz+d)^{2h}\,(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}\,\mathcal{O}_{h,\bar{h}}(z,\bar{z})$$

Holography: CFT operator $\mathcal{O}_{h,\bar{h}}$

with:

$h + \bar{h} = \Delta$ dimension

$h - \bar{h} = J$ spin

$$(h,\bar{h}) = \frac{1}{2}(\Delta + J, \Delta - J)$$

n-point amplitude on celestial sphere

$$\mathscr{A}(\{p_i, \epsilon_j\}) = i(2\pi)^4\ \delta^{(4)}\left(p_1 + p_2 - \sum_{k=3}^{n} p_k\right)\ A(\{p_i, \epsilon_j\})$$

with:

$$\langle ij\rangle = 2\ (\omega_i\omega_j)^{1/2}\ (z_i - z_j)$$
$$[ij] = 2\ (\omega_i\omega_j)^{1/2}\ (\bar{z}_i - \bar{z}_j)$$

$$\epsilon^\mu(q)_\pm = \frac{1}{\sqrt{2}}\begin{cases}\partial_z q^\mu = (\bar{z}, 1, -i, -\bar{z})\\ \partial_{\bar{z}} q^\mu = (z, 1, \ i, -z)\end{cases}$$

Celestial amplitudes $\widetilde{\mathscr{A}}$ of massless particles are obtained from momentum-space amplitudes $\mathscr{A}$ by Mellin transforms w.r.t. particle energies $\Delta_j = 1 + i\lambda_j$

$$\left\langle \prod_{k=1}^{n} \mathscr{O}_{\Delta_k, J_k}(z_k, \bar{z}_k)\right\rangle =$$

$$= \widetilde{\mathscr{A}}_{\{\Delta_k, J_k\}}(z_k, \bar{z}_k) = \left(\prod_{k=1}^{n}\int_0^\infty \omega_k^{\Delta_k - 1}\ d\omega_l\right)\ \delta^{(4)}(\omega_1 q_1 + \omega_2 q_2 - \sum_{m=3}^{n}\omega_m q_m) \times A(\omega_n, z_n, \bar{z}_n)$$

D=2 CFT correlators involve conformal wave packets

Gauge Amplitudes

example four-gluon amplitude:

$$\widetilde{\mathscr{A}}_4(-,-,+,+) = 8\pi\,\delta(r-\bar{r})\,\theta(r-1)\left(\prod_{i<j}^{4} z_{ij}^{\frac{h}{3}-h_i-h_j}\,\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}\right)$$
$$\times\, r^{\frac{5}{3}}\,(r-1)^{\frac{2}{3}}\,\delta\left(-4+\sum_{i=1}^{4}\Delta_i\right)$$

$$r = \frac{z_{12}\,z_{34}}{z_{23}\,z_{41}}$$

conformal invariant cross-ratio on CS^2

Pasterski, Shao, Strominger (2017)

$$h_1 = \tfrac{i}{2}\lambda_1,\;\; h_2 = \tfrac{i}{2}\lambda_2,\;\; h_3 = 1+\tfrac{i}{2}\lambda_3\,,h_4 = 1+\tfrac{i}{2}\lambda_4$$
$$\bar{h}_1 = 1+\tfrac{i}{2}\lambda_1,\;\; \bar{h}_2 = 1+\tfrac{i}{2}\lambda_2,\;\; \bar{h}_3 = \tfrac{i}{2}\lambda_3\,,\bar{h}_4 = \tfrac{i}{2}\lambda_4$$

$(12 \rightleftharpoons 34)_4,\quad r > 1$

$(13 \rightleftharpoons 24)_4,\quad 0 < r < 1$

$(14 \rightleftharpoons 23)_4,\quad r < 0$

higher-point: involve Gaussian hypergeometric functions like string amplitudes

Schreiber, Volovich, Zlotnikov (2017)

Celestial Conformal Field Theory (CCFT)

understand the nature of 2D CFT on celestial sphere, i.e. spectrum of fields and their interactions

- states, spectrum
- operator products (OPEs)
- energy momentum tensor, Virasoro algebra
- radial quantisation?
- conformal block expansion
- crossing symmetry and conformal bootstrap
- ⋮

recent progress

Operator product expansion

Gauge theory → collinear limits → singularities

$\overrightarrow{p_1} \parallel \overrightarrow{p_2}$

+ soft theorems determine almost uniquely scattering amplitudes

work by too many to cite properly

$$p^\mu = \omega q^\mu(z, \bar{z})$$

$z_1 \to z_2$

CFT → Operators approach → singularities OPE

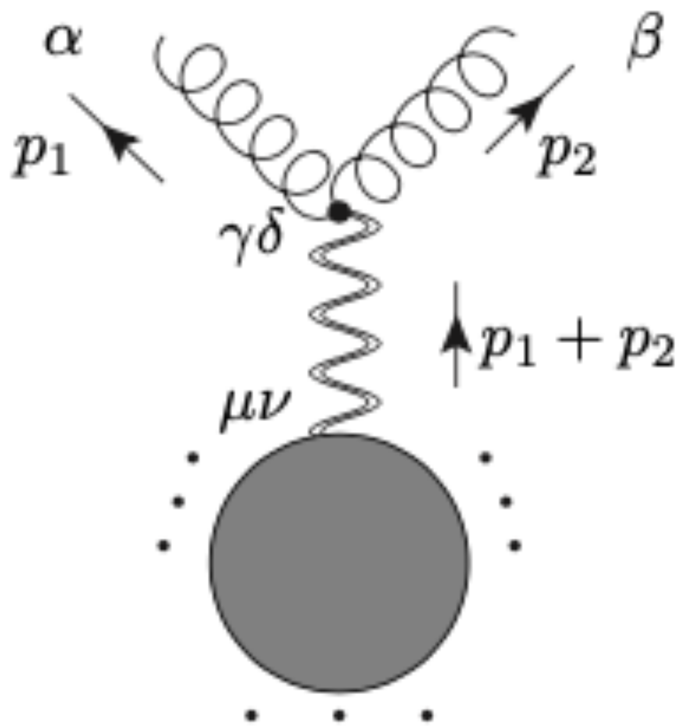
OPE for Conformal primaries

$$\mathcal{O}_i(z_i)\mathcal{O}_j(z_j) \sim \frac{C_{ijk}}{(z_i - z_j)^{h_i+h_j-h_k}}\mathcal{O}_k(z_k) + \dots$$

in 2D: structure constants
+Virasoro (local conformal) symmetry → CFT correlators

OPE in CCFT: Collinear singularities (2)

EYM Feynman Diagram for collinear gauge boson singularity



$$S_g^{\mu\nu} = \epsilon_1^{\alpha}\epsilon_2^{\beta} D^{\mu\nu}{}_{\gamma\delta}(p_1 + p_2) V^{\gamma\delta}_{\alpha,\beta}(p_1, p_2)$$

Fan, A.F. .Stieberger., Taylor, Zhu (2019)

gauge collinear

$$\mathcal{M}(1^+, 2^-, 3, \ldots, N) \sim \frac{1}{\bar{z}_{12}} \frac{\omega_1}{\omega_2 \omega_P} \mathcal{M}(P^+, 3, \ldots, N) + \frac{1}{z_{12}} \frac{\omega_2}{\omega_1 \omega_P} \mathcal{M}(P^-, 3, \ldots, N)$$

$$- \frac{z_{12}}{\bar{z}_{12}} \frac{\omega_1^2}{\omega_P^2} \mathcal{M}(P^{++}, 3, \ldots, N) - \frac{\bar{z}_{12}}{z_{12}} \frac{\omega_2^2}{\omega_P^2} \mathcal{M}(P^{--}, 3, \ldots, N)$$

$\omega_P = \omega_1 + \omega_2$

gravity collinear

Mellin transform → CCFT OPE!

Operator product expansion

Celestial conformal field theory (CCFT)

$$
\begin{aligned}
\mathcal{O}^a_{\Delta_1,-1}(z,\bar z)\mathcal{O}^b_{\Delta_2,+1}(w,\bar w) = &\frac{C_{(-,+)-}(\Delta_1,\Delta_2)}{z-w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),-1}(w,\bar w)\\
&+\frac{C_{(-+)+}(\Delta_1,\Delta_2)}{\bar z-\bar w}\sum_c f^{abc}\ \mathcal{O}^c_{(\Delta_1+\Delta_2-1),+1}(w,\bar w)\\
&+C_{(-+)--}(\Delta_1,\Delta_2)\,\frac{\bar z-\bar w}{z-w}\,\delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),-2}(w,\bar w)\\
&+C_{(-+)++}(\Delta_1,\Delta_2)\,\frac{z-w}{\bar z-\bar w}\,\delta^{ab}\,\mathcal{O}_{(\Delta_1+\Delta_2),+2}(w,\bar w)\ + reg\,.
\end{aligned}
$$

Derive from collinear limits of D=4 EYM amplitudes

Fan, Fotopoulos, St.St., Taylor, Zhu (2019)

D=4 S-matrix constrains OPE
or vice versa

Derive from first principles and consistency conditions

Pate, Raclariu, Strominger, Yuan (2019)

} extended BMS symmetry

Symmetries

At null infinity $\mathscr{I}^{\pm}$ more (hidden) symmetries present to constrain S-matrix

⟶ non-trivial consistency on amplitudes

$$z_i \to \frac{az_i + b}{cz_i + d}$$

$$SL(2,\mathbf{C})_{z_i}: \quad \widetilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow (cz_i + d)^{\Delta_i + J_i} \, (\bar{c}\bar{z}_i + \bar{d})^{\Delta_i - J_i} \, \widetilde{\mathscr{A}}_n(\{\Delta_i, J_i\})$$

$$P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2} = P^0 + P^3$$

St.Stieberger, Taylor (2018)

$$P^{(j)}_{-1/2,-1/2}: \quad \widetilde{\mathscr{A}}_n(\{\Delta_i, J_i\}) \longrightarrow \widetilde{\mathscr{A}}_n(\{\Delta_j + 1, J_i\})$$

translation operator P^μ *shifts conformal dimension* Δ_j

celestial gravitational amplitudes appear as gauge amplitudes translated in space-time

Symmetries of CCFT and Soft theorems

In usual QFT soft theorems $\omega_s \to 0$ play an important role
in consistency conditions on scattering amplitudes

$$\mathscr{M}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\, S_{\rm G}^{(0)}} + \underbrace{\omega_s^{0}\, S_{\rm G}^{(1)}} + \underbrace{\omega_s\, S_{\rm G}^{(2)}} + \ldots \Big)\, \mathscr{M}_n$$

on CS^2: poles at: $\Delta \to 1$ $\quad\Delta \to 0$ $\quad\Delta \to -1$

Weinberg (1965)
Cachazo, Strominger (2014)

$$\mathscr{A}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\, S_{\rm YM}^{(0)}} + \underbrace{\omega_s^{0}\, S_{\rm YM}^{(1)}} + \ldots \Big)\, \mathscr{A}_n$$

poles at: $\Delta \to 1$ $\quad\Delta \to 0$

soft theorems imply
Ward identities for asymptotic symmetries

Strominger (2013)

Mellin amplitude Residues: $\Delta \to 0,1,\ldots$ (conformally soft operators)

Kapec, Mitra, Raclariu, Strominger (2016)
Donnay, Puhm, Strominger (2018)

on CS^2: $\omega_s \to 0$

$$\mathscr{M}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\, S_{\rm G}^{(0)}}_{\Delta_s \to 1} + \underbrace{\omega_s^{0}\, S_{\rm G}^{(1)}}_{\Delta_s \to 0} + \underbrace{\omega_s\, S_{\rm G}^{(2)}}_{\Delta_s \to -1} + \dots \Big)\, \mathscr{M}_n$$

$$\mathscr{A}_{n+1} \longrightarrow \Big(\underbrace{\omega_s^{-1}\, S_{\rm YM}^{(0)}}_{\Delta_s \to 1} + \underbrace{\omega_s^{0}\, S_{\rm YM}^{(1)}}_{\Delta_s \to 0} + \dots \Big)\, \mathscr{A}_n$$

Cf.:

$$\int_0^\infty d\omega_s\, \omega_s^{\Delta_s - 1}\, \frac{e^{-J\omega_s}}{\omega_s} = \frac{1}{\Delta_s - 1} - \frac{J}{\Delta_s} + \frac{1}{2}\frac{J^2}{\Delta_s + 1} + \dots$$

typical IR poles in Mellin transform

E.g.: Yang-Mills

$$\mathscr{A}_{n+1} = \omega_s^{-1}\, \frac{z_{n1}}{z_{ns} z_{s1}} + \omega_s^{0} \left\{ \frac{1}{\omega_1}\frac{1}{z_{s1}} \left(\bar{z}_{s1}\partial_{\bar{z}_1} - 2\bar{h}_1 \right) + \frac{1}{\omega_n}\frac{1}{z_{ns}} \left(\bar{z}_{sn}\partial_{\bar{z}_n} - 2\bar{h}_n \right) \right\}^{s = n+1}$$

$$\times A_n(\{z_1, \bar{z}_1, \omega_1, J_1\}, \dots, \{z_n, \bar{z}_n, \omega_n, J_n\}) + \dots$$

Ward identities and BMS symmetries:

CCFT description of soft operators

conformally soft-graviton
$\Delta \to 0$

energy-momentum tensor T(z):

$$T(z) := \tilde{\mathcal{O}}_{\Delta=2,J=+2}(z,\bar{z}) = \frac{3}{\pi}\int d^2w \, \frac{\mathcal{O}_{\Delta=0,J=-2}(w,\bar{w})}{(z-w)^4}$$

$(h,\bar{h}) = (2,0)$

Kapec, Mitra, Raclariu, Strominger (2016)
Cheung, de La Fuente, Sundrum (2016)

shadow transformation:

$$\tilde{\mathcal{O}}^a_{\tilde{\Delta},\tilde{J}}(z,\bar{z}) = \tilde{\mathcal{O}}^a_{2-\Delta,-J}(z,\bar{z}) = \frac{(\Delta+J-1)}{\pi}\int_{\mathbf{C}} \frac{d^2w}{(z-w)^{2-\Delta-J}(\bar{z}-\bar{w})^{2-\Delta+J}} \, \mathcal{O}^a_{\Delta,J}(w,\bar{w})$$

Ferrara, Grillo, Parisi, Gatto (1972)
Dolan, Osborn (2012)

a) Single Soft limit:

A.F., T.R. Taylor (2019)

$$\langle T(z) \prod_{i=1}^{n} O_{\Delta_i,J_i}(z_i,\bar{z}_i)\rangle = \sum_{i=1}^{n}\left(\frac{h_{O_i}}{(z-z_i)^2}+\frac{\partial_{z_i}}{z-z_i}\right)\langle\prod_{i=1}^{n} O_{\Delta_i,J_i}(z_i,\bar{z}_i)\rangle$$

OPE with Primaries: $T(z)\mathcal{O}^{h_i,\bar{h}_i}(w) \sim \frac{h_i}{(z-w)^2}\mathcal{O}^{h_i,\bar{h}_i}(w) + \frac{1}{z-w}\partial_w\mathcal{O}^{h_i,\bar{h}_i}(w)$

Gauge boson/ graviton CCFT operators → Virasoro primaries

b) Double soft limits

$$\left\langle T(w)T(z)\prod_{i=2}^{n} \mathcal{O}_{\Delta_i,J_i}(z_i,\bar{z}_i)\right\rangle \sim \left\langle\left(\frac{2}{(z-w)^2}T(w)+\frac{1}{z-w}\partial_w T(w)\right)\prod_{i=2}^{n} O_{\Delta_i,J_i}(z_i,\bar{z}_i)\right\rangle$$

OPE:

$$T(z)T(w) = \frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{z-w}+\dots$$

$$T(w)\overline{T}(\bar{z}) = reg\,.$$

$\underline{c=0}$

A.F ,.Stieberger., Taylor, Zhu (2019)

(ii) supertranslation operator P(z):

conformally soft-graviton
$\Delta \to 1$

$$P(z,\bar{z}) := \partial_{\bar{z}} \mathcal{O}_{\Delta=1,J=+2}(z,\bar{z}) \qquad (h,\bar{h}) = (\tfrac{3}{2},\tfrac{1}{2})$$

$$\left\langle P(z_0) \prod_{j=1}^{n} \mathcal{O}_{\Delta_j,J_j}(z_j,\bar{z}_j) \right\rangle \sim \sum_{i=1}^{n} \frac{1}{z_0 - z_i} \left\langle \prod_{n=1}^{n} \mathcal{O}_{\Delta_j,J_j}(z_j,\bar{z}_j) \right\rangle \Bigg|_{\Delta_i \to \Delta_i + 1}$$

Adamo, Mason, Sharma/Guevara/ Puhm (2019)

$$P(z)\mathcal{O}_{\Delta,J}(w,\bar{w}) \sim \frac{1}{z-w} \mathcal{O}_{\Delta+1,J}(w,\bar{w}) + \ reg\,.$$

From double soft limits

OPEs:

$$T(z)P(w) = \frac{3}{2(z-w)^2} P(w) + \frac{1}{z-w} \partial_w P(w) + reg\,.$$

$$\bar{T}(\bar{z})P(w) = -\frac{\mathcal{O}_{1,+2}(w,\bar{w})}{(\bar{z}-\bar{w})^3} + \frac{1}{2(\bar{z}-\bar{w})^2} P(w) + \frac{1}{\bar{w}-\bar{z}} \partial_{\bar{w}} P(w) + \ reg\,.$$

Transforms as an antiholomorphic descendant

$$P(z)P(w) \sim reg\,.$$

Use OPEs to extract symmetry algebra

Virasoro generators $$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$$

Primary field mode expansion: $$\mathcal{O}^{h,\bar{h}}(z,\bar{z}) = \sum_{m,n} \mathcal{O}^{h,\bar{h}}_{m,n}\, z^{-m-h} \bar{z}^{-n-\bar{h}}$$

Translation operators: $$\begin{pmatrix} P_{-\frac{1}{2},-\frac{1}{2}} = P_0 + P_3 = e^{(\partial_h + \partial_{\bar{h}})/2} & P_{-\frac{1}{2},\frac{1}{2}} = P_1 - iP_2 = \bar{z} e^{(\partial_h + \partial_{\bar{h}})/2} \\ P_{\frac{1}{2},-\frac{1}{2}} = P_1 + iP_2 = z e^{(\partial_h + \partial_{\bar{h}})/2} & P_{\frac{1}{2},\frac{1}{2}} = P_0 - P_3 = z\bar{z} e^{(\partial_h + \partial_{\bar{h}})/2} \end{pmatrix}$$

Define operator : $$P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2} = P^0 + P^3$$

then

$$\left[P_{-\frac{1}{2},-\frac{1}{2}}, \mathcal{O}_{h,\bar{h}}(z,\bar{z})\right] \to \left[P_{-\frac{1}{2},-\frac{1}{2}}, \mathcal{O}^{h,\bar{h}}_{m,n}\right] = \mathcal{O}^{h+\frac{1}{2},\bar{h}+\frac{1}{2}}_{m-\frac{1}{2},n-\frac{1}{2}}$$

generates flow in primary field mode expansion

In addition to Virasoro symmetry, we construct
all supertranslation generators acting on primary fields

A.F., Stieberger, Taylor, Zhu (2019)

construct:

$$P_{n-\frac{1}{2},-\frac{1}{2}} = \frac{1}{i\pi(n+1)} \oint dw\, w^{n+1} \left[T(w), P_{-\frac{1}{2},-\frac{1}{2}}\right]$$

$$P_{n-\frac{1}{2},m-\frac{1}{2}} = \frac{1}{i\pi(m+1)} \oint d\bar{w}\, \bar{w}^{m+1} \left[\bar{T}(\bar{w}), P_{n-\frac{1}{2},-\frac{1}{2}}\right]$$

$$P_{-1/2,-1/2} = e^{(\partial_h+\partial_{\bar{h}})/2}$$

we find:

$$\left[P_{n-\frac{1}{2},m-\frac{1}{2}}, \mathcal{O}_{h,\bar{h}}(z,\bar{z})\right] = z^n \bar{z}^m \mathcal{O}_{h+\frac{1}{2},\bar{h}+\frac{1}{2}}(z,\bar{z})$$

→ $P_{k,l}, \bar{P}_{k,l}$

→ local (or extended) BMS algebra:

$$[P_{ij}, P_{k,l}] = 0\ ,$$

$$[L_n, P_{k,l}] = \left(\frac{1}{2}n - k\right) P_{n+k,l} + n(n^2-1)\ C_{n,k}\ ,$$

$$[\bar{L}_n, P_{k,l}] = \left(\frac{1}{2}n - l\right) P_{k,n+l} + n(n^2-1)\ \bar{C}_{n,l}\ .$$

$m, n \in \mathbf{Z},\ i, j, k, l \in \mathbf{Z}+\frac{1}{2}$

field dependent central charges =0

Bondi, Burg, Metzner (1962)
Sachs (1962)
Barnich Troessaert (2010)
Barnich (2017)

Conformal soft-theorems ⟷ Ward identities ⟷ extended BMS algebra

Extended BMS group on celestial sphere

global BMS symmetry on celestial sphere	local BMS symmetry on celestial sphere
<u>Lorentz group:</u> global conformal transformations on celestial sphere SL(2,C) $z \to \frac{az+b}{cz+d}$ $\quad L_{-1} = \partial$, $\quad L_0 = z\partial + h$, $\quad L_1 = z^2\partial + 2hz$	local conformal transformations = superrotations T(z) $[L_m, L_n] = (m-n)\, L_{m+n}$ $[\bar{L}_m, \bar{L}_n] = (m-n)\, \bar{L}_{m+n}$
<u>global space-time translation:</u> Abelian subgroup of supertranslations $P_{-1/2,-1/2} = e^{(\partial_h + \partial_{\bar{h}})/2}$ $\quad P_{1/2,1/2} = z\ e^{(\partial_h + \partial_{\bar{h}})/2}$ $P_{-1/2,1/2} = \bar{z}\ e^{(\partial_h + \partial_{\bar{h}})/2}$ $\quad P_{-1/2,1/2} = \lvert z \rvert^2\ e^{(\partial_h + \partial_{\bar{h}})/2}$	local space-time translations =supertranslations P(z) $P_{n-\frac{1}{2},m-\frac{1}{2}} \quad n, m \in \mathbf{Z}$

→ Symmetries of the celestial OPEs and correlators
S-matrix (non-trivial consistency)

Supersymmetric Extended BMS

Supermultiplets of Conformal Primary Wavefunctions

Scalar CPW $\varphi^{\pm}_{\Delta}(X^\mu, z, \bar{z}) = \int_0^\infty d\omega\, \omega^{\Delta-1} e^{\pm i\omega q\cdot X - \epsilon\omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(-q\cdot X \mp i\epsilon)^\Delta}$

$(h, \bar{h}) = (\frac{\Delta}{2}, \frac{\Delta}{2})$

Fermion CPW $\psi^{\pm}_{\Delta,\alpha}(X, z, \bar{z}) = |q\rangle_\alpha \int d\omega \omega^{\Delta+\frac{1}{2}-1} e^{\pm i\omega q\cdot X - \epsilon\omega} = |q\rangle_\alpha\ \varphi^{\pm}_{\Delta+\frac{1}{2}}(X, z, \bar{z})$

$(h, \bar{h}) = (\frac{\Delta}{2} - \frac{1}{4}, \frac{\Delta}{2} + \frac{1}{4})$

Solve Weyl equation $\bar{\sigma}^\mu \partial_\mu \psi_\Delta = 0$

$$|p\rangle_\alpha = \sqrt{\omega}\begin{pmatrix} z \\ 1 \end{pmatrix} = \sqrt{\omega}\,|q\rangle_\alpha \qquad [p|_{\dot\alpha} = \sqrt{\omega}\begin{pmatrix} \bar{z} \\ 1 \end{pmatrix} = \sqrt{\omega}[q|_{\dot\alpha}$$

$$\langle p|^\alpha = \sqrt{\omega}\begin{pmatrix} 1 \\ -z \end{pmatrix} = \sqrt{\omega}\langle q|^\alpha \qquad |p]^{\dot\alpha} = \sqrt{\omega}\begin{pmatrix} 1 \\ -\bar{z} \end{pmatrix} = \sqrt{\omega}\,|q]^{\dot\alpha}\,.$$

$|q\rangle : \ (-\frac{1}{2}, 0) \qquad [q| : \ (0, -\frac{1}{2})$

Dirac Spinors $\Psi^{\pm}_{\Delta,\ell=-\frac{1}{2}}(X, z, \bar{z}) = \begin{pmatrix} \psi^{\pm}_{\Delta,\alpha} \\ 0 \end{pmatrix}, \qquad \Psi^{\pm}_{\Delta,\ell=+\frac{1}{2}}(X, z, \bar{z}) = \begin{pmatrix} 0 \\ \bar{\chi}^{\pm\dot\alpha}_{\Delta} \end{pmatrix}$

Orthonormal under Dirac inner product

Quantum Fields 4D → expand in CPW

$$\varphi(X) = \int d^2z \; d(i\Delta) \left[a_{\Delta+}(z)\varphi^{-}_{\Delta^*}(X,z) + a^{\dagger}_{\Delta-}(z)\varphi^{+}_{\Delta}(X,z) \right],$$

$$\psi_\alpha(X) = \int d^2z \; d(i\Delta) \left[b_{\Delta+}(z)\psi^{-}_{\Delta^*,\alpha}(X,z) + b^{\dagger}_{\Delta-}(z)\psi^{+}_{\Delta,\alpha}(X,z) \right].$$

$$a_{\Delta\pm}(z) = \int_0^\infty d\omega \, \omega^{\Delta-1} \, a_\pm(\overrightarrow{p}) \,, \qquad b_{\Delta\pm}(z) = \int_0^\infty d\omega \, \omega^{\Delta-1} \, b_\pm(\overrightarrow{p})$$

$$(h, \bar{h}) = (\frac{\Delta}{2} + \frac{1}{4}, \frac{\Delta}{2} - \frac{1}{4})$$

Oscillator Algebra

$$[a_{\Delta\pm}(z), a^{\dagger}_{\Delta'\pm}(z')] = 8\pi^4 \delta(\Delta + (\Delta')^* - 2)\delta^{(2)}(z - z'),$$

$$\{b_{\lambda\pm}(z), b^{\dagger}_{\lambda'\pm}(z')\} = 8\pi^4 \delta(\Delta + (\Delta')^* - 2)\delta^{(2)}(z - z').$$

Celestial Holography

$$a_{\Delta\pm}, b_{\Delta\pm} \mapsto \mathcal{O}_{\Delta,J}(z,\bar{z}) \,, \qquad |0\rangle_{D=4} \mapsto |0\rangle_{CS_2} \,.$$

Repeat for gauge and graviton multiplet

i.e gauge CPW: $v^{\mu\pm}_{\Delta,J}(X,z,\bar{z}) = \epsilon^{\mu}_{J}(q,r)\varphi^{\pm}_{\Delta}(X,z,\bar{z})$

↓ Susy

$$\psi^{\pm}_{\Delta}(X,z,\bar{z}) = \frac{1}{\sqrt{2}} e^{\frac{\partial_h + \partial_{\bar{h}}}{4}} v^{\mu\pm}_{\Delta,J=-1}(X,z,\bar{z})\sigma_{\mu}|q]$$

$$\bar{\psi}^{\pm}_{\Delta}(X,z,\bar{z}) = \frac{1}{\sqrt{2}} e^{\frac{\partial_h + \partial_{\bar{h}}}{4}} v^{\mu\pm}_{\Delta,J=+1}(X,z,\bar{z})\bar{\sigma}_{\mu}|q\rangle$$

In CCFT we find

$J^c = J - \frac{1}{2}$ *restricted by multiplet content*

$$[\langle \eta Q\rangle, \mathcal{O}_{\Delta,J^c}] = \langle \eta q\rangle \mathcal{O}_{(\Delta+\frac{1}{2}),J}$$

$$[[\bar{\eta}\bar{Q}], \mathcal{O}_{\Delta,J}] = [\bar{\eta}q]\mathcal{O}_{(\Delta+\frac{1}{2}),J^c}$$

supermutliplet	J	J^c
chiral	0, +1/2	-1/2,0
gauge	-1/2,+1	-1,+1/2
gravitational	-3/2, +2	-2, +3/2

CCFT Supersymmetry currents

use shadow transform

$$S(z) = \lim_{\Delta \to \frac{1}{2}} \frac{\Delta - \frac{1}{2}}{\pi} \int d^2 z' \frac{1}{(z - z')^3} \mathcal{O}_{\Delta, -\frac{3}{2}}(z', \bar{z}') \qquad (h, \bar{h}) = (\frac{3}{2}, 0)$$

$$\bar{S}(\bar{z}) = \lim_{\Delta \to \frac{1}{2}} \frac{\Delta - \frac{1}{2}}{\pi} \int d^2 z' \frac{1}{(\bar{z} - \bar{z}')^3} \mathcal{O}_{\Delta, +\frac{3}{2}}(z', \bar{z}') \qquad (h, \bar{h}) = (0, \frac{3}{2})$$

apply on leading soft gravitino theorem

$$S(z) \mathcal{O}_{\Delta, J^c}(w, \bar{w}) = \frac{1}{z - w} \mathcal{O}_{\Delta + \frac{1}{2}, J}(w, \bar{w}) \ + \ regular$$

$$\bar{S}(\bar{z}) \mathcal{O}_{\Delta, J}(w, \bar{w}) = \frac{1}{\bar{z} - \bar{w}} \mathcal{O}_{\Delta + \frac{1}{2}, J^c}(w, \bar{w}) \ + \ regular$$

Super BMS algebra

Laurent expansion of fields

$$S(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{G_n}{z^{n+\frac{3}{2}}}\,, \qquad G_n = \oint dz\, z^{n+1/2}\, S(z)$$

$$\bar{S}(\bar{z}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\bar{G}_n}{z^{n+\frac{3}{2}}}\,, \qquad \bar{G}_n = \oint d\bar{z}\, \bar{z}^{n+1/2}\, \bar{S}(\bar{z})$$

$$S(z)\mathcal{O}_{\Delta,J^c}(w,\bar{w}) = \frac{1}{z-w}\mathcal{O}_{\Delta+\frac{1}{2},J}(w,\bar{w}) \;+\; regular$$

$$\bar{S}(\bar{z})\mathcal{O}_{\Delta,J}(w,\bar{w}) = \frac{1}{\bar{z}-\bar{w}}\mathcal{O}_{\Delta+\frac{1}{2},J^c}(w,\bar{w}) \;+\; regular$$

OPEs with primaries imply

$$[G_n, \mathcal{O}_{\Delta,J^c}(w,\bar{w})] = w^{n+1/2}\,\mathcal{O}_{\Delta+\frac{1}{2},J}(w,\bar{w})$$

$$[\bar{G}_n, \mathcal{O}_{\Delta,J}(w,\bar{w})] = \bar{w}^{n+1/2}\,\mathcal{O}_{\Delta+\frac{1}{2},J^c}(w,\bar{w})$$

From action of global Susy on Operators we identify

$$[\langle \eta Q\rangle, \mathcal{O}_{\Delta,J^c}] = \langle \eta q\rangle \mathcal{O}_{(\Delta+\frac{1}{2}),J}$$

$$[[\bar{\eta}\bar{Q}], \mathcal{O}_{\Delta,J}] = [\bar{\eta} q]\mathcal{O}_{(\Delta+\frac{1}{2}),J^c}$$

$$Q_1 \to G_{+1/2}\,, \qquad Q_2 \to G_{-1/2}\,,$$

$$\bar{Q}_{\dot{1}} \to \bar{G}_{+1/2}\,, \qquad \bar{Q}_{\dot{2}} \to \bar{G}_{-1/2}\,.$$

Extended Super BMS Algebra

Laurent expansion of fields:

$$S(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{G_n}{z^{n+\frac{3}{2}}}, \qquad G_n = \oint dz\, z^{n+1/2}\, S(z)$$

$$\bar{S}(\bar{z}) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{\bar{G}_n}{\bar{z}^{n+\frac{3}{2}}}, \qquad \bar{G}_n = \oint d\bar{z}\, \bar{z}^{n+1/2}\, \bar{S}(\bar{z})$$

$G_{\pm\frac{1}{2}}, \bar{G}_{\pm\frac{1}{2}}$ related to SUSY generators in D=4

Recall: Supertranslation operator: $\mathscr{P}(z,\bar{z}) \equiv \sum_{n,m\in\mathbf{Z}} P_{n-\frac{1}{2},m-\frac{1}{2}}\, z^{-n-1}\bar{z}^{-m-1} = \underbrace{S(z)\bar{S}(\bar{z}) + \bar{S}(\bar{z})S(z)}_{\Delta \to \frac{1}{2}}$

proceed as for the bosonic case extract the Super BMS algebra

$$\{G_m, \bar{G}_n\} = P_{m,n}$$

$$\{G_m, G_n\} = \{\bar{G}_m, \bar{G}_n\} = 0$$

$$[P_{k,l}, G_n] = [P_{k,l}, \bar{G}_m] = 0$$

$$[L_m, G_k] = \left(\frac{1}{2}m - k\right) G_{m+k}$$

$$[\bar{L}_m, \bar{G}_l] = \left(\frac{1}{2}m - l\right) \bar{G}_{m+l}$$

$$[L_m, \bar{G}_n] = [\bar{L}_m, G_n] = 0$$

$m, n \in \mathbf{Z},\ i, j, k, l \in \mathbf{Z}+\frac{1}{2}$

superrotations L_n $\quad \Delta \to 0$

supertranslations $P_{k,l}$ $\quad \Delta \to 1$

supersymmetry G_k $\quad \Delta \to \frac{1}{2}$

D=4 SUSY/SUGRA chiral and gauge multiplets → supersymmetric generalization of BMS symmetry

N=1 supersymmetric extension of BMS algebra on CS^2

Radial Quantization in CCFT

Radial quantization of 2D conformal field theory (CFT)- the asymptotic states characterized by dimensions Δ and spin J are created by acting on the vacuum state with primary quantum field operators:

$$|\Delta, J; in\rangle = \lim_{z,\bar{z}\to 0} \phi_{\Delta,J}(z,\bar{z})|0\rangle \;, \qquad \langle \Delta, J; out| = \lim_{z,\bar{z}\to 0} \langle 0|\phi^{\dagger}_{\Delta,J}(z,\bar{z})$$

$$[\phi_{\Delta,J}(z,\bar{z})]^{\dagger} = \bar{z}^{-2h} z^{-2\bar{h}} \phi_{\Delta,J}(1/\bar{z}, 1/z)$$

Inner product

$$\langle \Delta, J; in|\Delta, J; out\rangle = \lim_{\xi,\bar{\xi}\to\infty} \bar{\xi}^{2h}\xi^{2\bar{h}} \langle 0|\phi_{\Delta,J}(\bar{\xi},\xi)\phi_{\Delta,J}(0,0)|0\rangle$$

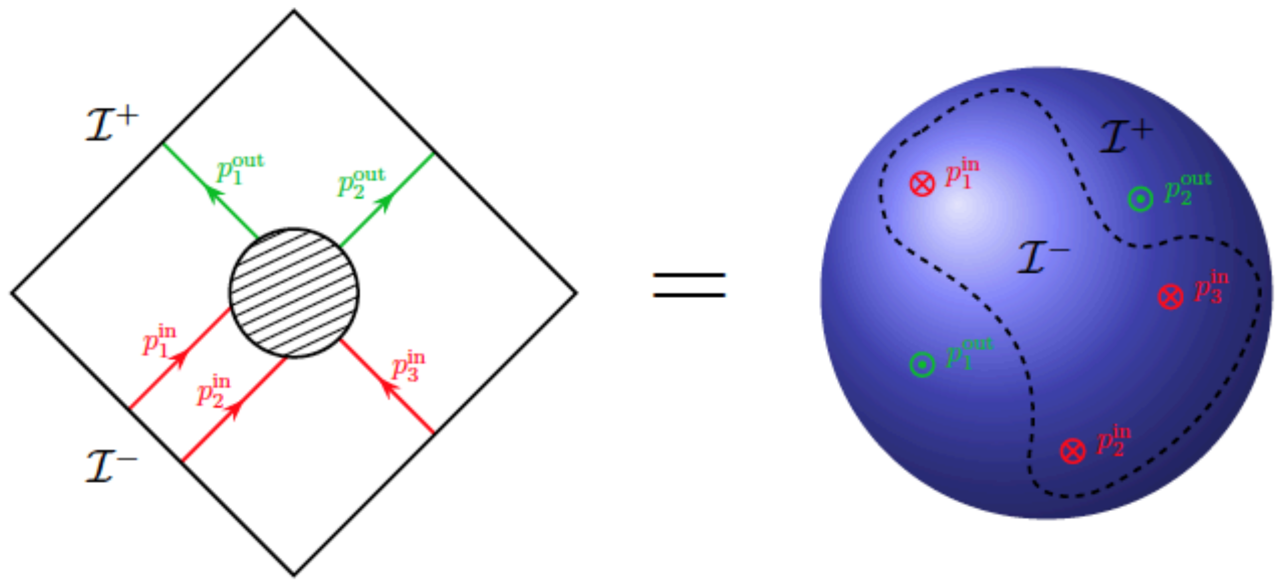
Conformally covariant 2pt function

$$\langle 0|\phi_{\Delta,J}(z,\bar{z})\phi_{\Delta,J}(w,\bar{w})|0\rangle = \frac{C}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}}$$

$$\downarrow$$

$$\langle \Delta, J; in|\Delta, J; out\rangle = C$$

Radial Quantization in CCFT



Due to antipodal identification between $\mathscr{I}^+$ and $\mathscr{I}^-$ free particles enter and exit at the same point on the celestial sphere

$$(\Phi_1, \Phi_2)_{KG} = -i \int_{\Sigma_3} d^3\Sigma^\mu \Phi_1 \overleftrightarrow{\partial_\mu} \Phi_2^*$$

From Klein-Gordon inner product, Mellin transform of 2-point function:

$$\langle 0 | \phi_{\Delta_2^*, -J}(z, \bar{z}) \phi_{\Delta_1, J}(w, \bar{w}) | 0 \rangle = \widetilde{C} k^{-1}_{1-h, 1-\bar{h}} \, \delta(\Delta_1 + \Delta_2^* - 2) \; \delta^{(2)}(z - w) \qquad (\Delta = 1 + i\mathbb{R})$$

Does not look like a standard CFT 2-point

Adjoint of Conformal generators $\quad (\Phi_1, L_n \Phi_2)_{KG} = -(\bar{L}_n \Phi_1, \Phi_2)_{KG} \longrightarrow L_n^\dagger = -\bar{L}_n$

Keep property of 4D: in/out conjugates

See recent Pasterski et al 2021
Strominger et al 2021

Radial Quantization in CCFT

Alternative

Shadow transform

$$\tilde{\phi}(z,\bar{z}) = k_{h,\bar{h}} \int d^2y (z-y)^{2h-2}(\bar{z}-\bar{y})^{2\bar{h}-2}\phi(y,\bar{y}) \ , \ k_{h,\bar{h}} = (-1)^{2(h-\bar{h})}\frac{\Gamma(2-2h)}{\pi\Gamma(2\bar{h}-1)}$$

Define BPZ-like Conjugate

$$[\phi_{\Delta,J}(z,\bar{z})]^{\dagger} = \bar{z}^{-2h}z^{-2\bar{h}}\widetilde{\phi_{\Delta^*,-J}}(1/\bar{z},1/z) = \bar{z}^{-2h}z^{-2\bar{h}}\tilde{\phi}_{\Delta,J}(1/\bar{z},1/z) \qquad (\Delta = 1+i\mathbb{R})$$

$$\langle 0|\tilde{\phi}_{\Delta,J}(z,\bar{z})\phi_{\Delta,J}(w,\bar{w})|0\rangle = \frac{\widetilde{C}}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}}, \qquad \langle \Delta, J; in|\Delta, J; out\rangle = \widetilde{C}$$

Global conformal algebra adjoints: $L_n^{\dagger} = L_{-n}, \quad n = 0,1,-1$
(shadow does not commute with full Virasoro)

Taking into account in and out states, we can propose the conjugate of an out as follows:

$$[\phi_{\Delta}^{(+)}(z,\bar{z})]^{\dagger,rad} = \bar{z}^{-2h}z^{-2\bar{h}}\widetilde{\phi}_{\Delta}^{(-)}(\frac{1}{\bar{z}},\frac{1}{z}) = \bar{z}^{-2h}z^{-2\bar{h}}\widetilde{\phi_{2-\Delta}^{(-)}}(\frac{1}{\bar{z}},\frac{1}{z})$$

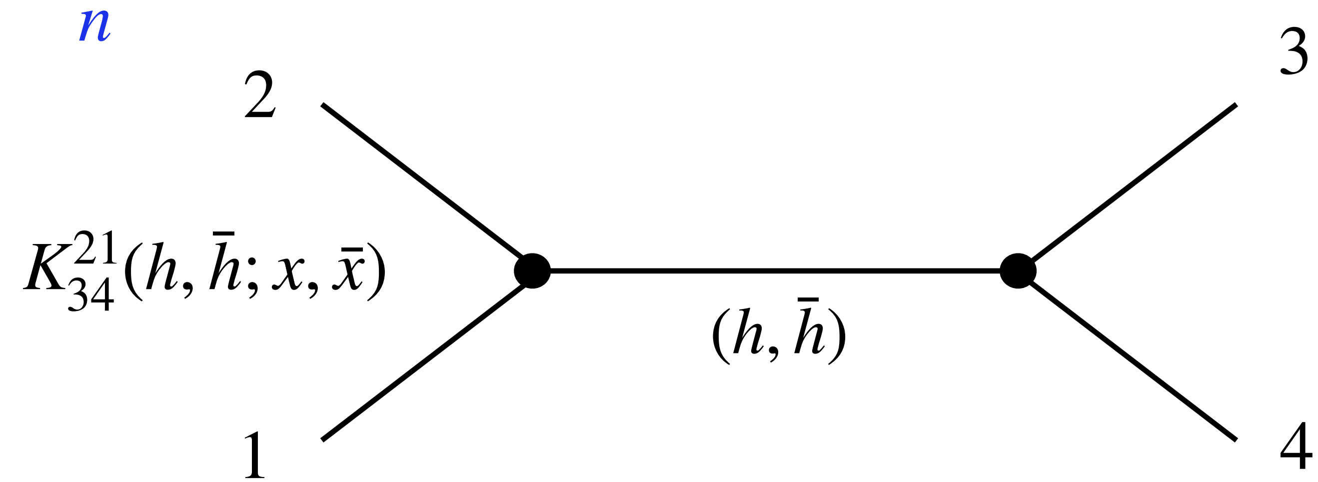
Conformal Blocks

standard CFT: *conformal block decomposition of correlation functions*
→ comprises full spectrum

$$G^{21}_{34}(x,\bar{x}) = \lim_{z_{1'},\bar{z}_{1'}\to\infty} z_{1'}^{2h_{1'}}\bar{z}_{1'}^{2\bar{h}_{1'}}\left\langle \tilde{\phi}^{a_1}_{\tilde{\Delta}_1,+}(z_{1'},\bar{z}_{1'})\phi^{a_2}_{\Delta_2,-}(1,1)\phi^{a_3}_{\Delta_3,+}(z'=x,\bar{z}'=\bar{x})\phi^{a_4}_{\Delta_4,+}(0,0)\right\rangle$$

$$\stackrel{!}{=} \sum_n C^n_{34}C^n_{12}\; A^{21}_{34}(n;x,\bar{x}) = \sum_n C^n_{34}C^n_{12}\; \mathcal{F}^{21}_{34}(n;x)\times\bar{\mathcal{F}}^{21}_{34}(n;\bar{x})$$

$$\equiv \sum_{h,\bar{h}} a_{h,\bar{h}}\; K^{21}_{34}(h,\bar{h};x,\bar{x})$$



Di Francesco, Mathieu, Senechal (1997)
Osborn (2012)

conformal block:

$$K^{21}_{34}(h,\bar{h};x,\bar{x}) = x^{h-h_3-h_4}\;{}_2F_1\left[\begin{matrix}h-h_{12},h+h_{34}\\2h\end{matrix};x\right]\times\;\bar{x}^{\bar{h}-\bar{h}_3-\bar{h}_4}{}_2F_1\left[\begin{matrix}\bar{h}-\bar{h}_{12},\bar{h}+\bar{h}_{34}\\2\bar{h}\end{matrix};\bar{x}\right]$$

Conformal Blocks in CCFT

$$\left\langle \phi^{a_1}_{\Delta_1,-}(z_1,\bar{z}_1)\phi^{a_2}_{\Delta_2,-}(z_2,\bar{z}_2)\phi^{a_3}_{\Delta_3,+}(z_3,\bar{z}_3)\phi^{a_4}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle = \delta(\sum_{i=1}^{4}\lambda_i)\prod_{i<j}(z_{ij}\bar{z}_{ij})^{-i\frac{\lambda_i}{2}-i\frac{\lambda_j}{2}}$$

$$\times\,\delta(z-\bar{z})\left(\frac{z_{12}}{z_{13}z_{24}z_{34}}\right)\left(\frac{\bar{z}_{34}^2}{\bar{z}_{13}\bar{z}_{24}\bar{z}_{14}\bar{z}_{23}}\right)(f^{a_1a_2b}f^{a_3a_4b}-zf^{a_1a_3b}f^{a_2a_4b})$$

$(12 \rightleftharpoons 34)_4, \quad z>1$

$(13 \rightleftharpoons 24)_4, \quad 0<z<1$

$(14 \rightleftharpoons 23)_4, \quad z<0$

Conformal cross-ratio

$$\left\langle \tilde{\phi}^{a_1}_{\tilde{\Delta}_1,+}(z_{1'},\bar{z}_{1'})\phi_{\Delta_2,-}(z_2,\bar{z}_2)\phi_{\Delta_3,+}(z_3,\bar{z}_3)\phi_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle =$$

$$\int\frac{d^2z_1}{(z_1-z_1')^{2-i\lambda_1}(\bar{z}_1-\bar{z}_1')^{-i\lambda_1}}\left\langle \phi^{a_1}_{\Delta_1,-}(z_1,\bar{z}_1)\phi^{a_2}_{\Delta_2,-}(z_2,\bar{z}_2)\phi^{a_3}_{\Delta_3,+}(z_3,\bar{z}_3)\phi^{a_4}_{\Delta_4,+}(z_4,\bar{z}_4)\right\rangle$$

$$x=z'=\frac{z_{1'2}z_{34}}{z_{1'3}z_{24}}\,,\qquad \bar{x}=\bar{z}'=\frac{\bar{z}_{1'2}\bar{z}_{34}}{\bar{z}_{1'3}\bar{z}_{24}}$$

Conformal Blocks in CCFT

$$G^{21}_{34}(x,\bar{x}) = \lim_{z_{1'},\bar{z}_{1'}\to\infty} z_{1'}^{2h_{1'}}\bar{z}_{1'}^{2\bar{h}_{1'}}\left\langle \tilde{\phi}^{a_1}_{\tilde{\Delta}_1,+}(z_{1'},\bar{z}_{1'})\phi^{a_2}_{\Delta_2,-}(1,1)\phi^{a_3}_{\Delta_3,+}(z'=x,\bar{z}'=\bar{x})\phi^{a_4}_{\Delta_4,+}(0{,}0)\right\rangle =$$

$$= (1-x)^{1+i\lambda_4}x^{-1-\frac{i\lambda_3}{2}-\frac{i\lambda_4}{2}}(1-\bar{x})^{-1+i\lambda_4}\bar{x}^{2-\frac{i\lambda_3}{2}-\frac{i\lambda_4}{2}}$$
$$\times\left[f^{a_1a_2b}f^{a_3a_4b}I(x)+f^{a_1a_3b}f^{a_2a_4b}\tilde{I}(x)\right].$$

Integrals I(x) and $\tilde{I}(x)$ can be expressed in terms of the Appell function F_1.
postpone this step and first consider the conformal soft limit $\Delta\to 1(\lambda_1\to 0)$.

The shadow current $\quad \tilde{\phi}^{a_1}_{\tilde{\Delta}_1=1,+}(z_{1'},\bar{z}_{1'}) = -2\pi\,\tilde{J}^{a_1}(z_{1'}),\quad h_1'=1,\ \bar{h}_1'=0$

generates global gauge group transformations

$$\left\langle \bar{J}^{a_1}(\bar{w})\phi^{a_2}_{\Delta_2,J_2}(z_2,\bar{z}_2)\cdots\phi^{a_N}_{\Delta_N,J_N}(z_N,\bar{z}_N)\right\rangle$$
$$=\sum_{i=2}^{N}\sum_b\frac{f^{a_1a_ib}}{\bar{w}-\bar{z}_i}\left\langle\phi^{a_2}_{\Delta_2,J_2}(z_2,\bar{z}_2)\cdots\phi^{b}_{\Delta_i,J_i}(z_i,\bar{z}_i)\cdots\phi^{a_N}_{\Delta_N,J_N}(z_N,\bar{z}_N)\right\rangle$$

$$\tilde{J}^a(z) = -\frac{1}{2\pi}\int\frac{d^2w}{(z-w)^2}\,\bar{J}^a(\bar{w})$$

Kac -Moody algebra

$$\left\langle \tilde{J}^{a_1}(z)\phi^{a_2}_{\Delta_2,J_2}(z_2,\bar{z}_2)\cdots\phi^{a_N}_{\Delta_N,J_N}(z_N,\bar{z}_N)\right\rangle$$
$$=\sum_{i=2}^{N}\sum_b\frac{f^{a_1a_ib}}{z-z_i}\left\langle\phi^{a_2}_{\Delta_2,J_2}(z_2,\bar{z}_2)\cdots\phi^{b}_{\Delta_i,J_i}(z_i,\bar{z}_i)\cdots\phi^{a_N}_{\Delta_N,J_N}(z_N,\bar{z}_N)\right\rangle.$$

Conformal Blocks in CCFT

$$I_s(x) = \frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{x^2\bar{x}} \int_1^{+\infty} dr \frac{r^{1-i\lambda_2}}{(r-1)^{1+i\lambda_4}} \left(\frac{r}{x} - 1\right)^{-2} = \frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{\bar{x}} B(1-i\lambda_3, -i\lambda_4)\, {}_2F_1\left(\begin{matrix} 2, 1-i\lambda_3 \\ 1+i\lambda_2 \end{matrix}; x\right)$$

$$I_t(x) = -\frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{x^2\bar{x}} \int_0^{1} dr \frac{r^{1-i\lambda_2}}{(1-r)^{1+i\lambda_4}} \left(\frac{r}{x} - 1\right)^{-2} = -\frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{x^2\bar{x}} B(2-i\lambda_2, -i\lambda_4)\, {}_2F_1\left(\begin{matrix} 2, 2-i\lambda_2 \\ 2+i\lambda_3 \end{matrix}; \frac{1}{x}\right)$$

$$I_u(x) = \frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{x^2\bar{x}} \int_0^{\infty} dr \frac{r^{1-i\lambda_2}}{(1+r)^{1+i\lambda_4}} \left(\frac{r}{x} + 1\right)^{-2} = \frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{\bar{x}} B(2-i\lambda_2, 1-i\lambda_3)\, {}_2F_1\left(\begin{matrix} 2, 1-i\lambda_3 \\ 3+i\lambda_4 \end{matrix}; 1-x\right)$$

$$G_{34}^{21}(x,\bar{x})_{s,\lambda_1=0} = (1-\bar{x})^{-1+i\lambda_4}\bar{x}^{1+i\lambda_2}\left[f^{a_1a_2b}f^{a_3a_4b}S_{34}^{21}(x) + f^{a_1a_3b}f^{a_2a_4b}\tilde{S}_{34}^{21}(x)\right]$$

$$S_{34}^{21}(x) = (1-x)^{1+i\lambda_4}x^{-1+i\lambda_2}{}_2F_1\left(\begin{matrix} 2, 1-i\lambda_3 \\ 1+i\lambda_2 \end{matrix}; x\right) B(1-i\lambda_3, -i\lambda_4)\ ,$$

$$\tilde{S}_{34}^{21}(x) = -(1-x)^{1+i\lambda_4}x^{-1+i\lambda_2}{}_2F_1\left(\begin{matrix} 2, -i\lambda_3 \\ i\lambda_2 \end{matrix}; x\right) B(-i\lambda_3, -i\lambda_4)$$

Conformal Blocks in CCFT

After some hypergeometric gymnastics

(or alpha-space see Hoqervorst, vRees, Rutter (2017.2020))

$$S^{21}_{34}(x) = \sum_{m=1}^{\infty} x^{h_m-h_3-h_4}\, a_m\, {}_2F_1\left(\begin{matrix} h_m-h_{12}, h_m+h_{34} \\ 2h_m \end{matrix}; x\right) ,$$

$$\tilde{S}^{21}_{34}(x) = \sum_{m=1}^{\infty} x^{h_m-h_3-h_4}\, \tilde{a}_m\, {}_2F_1\left(\begin{matrix} h_m-h_{12}, h_m+h_{34} \\ 2h_m \end{matrix}; x\right) ,$$

$$h_m = m + \frac{i\lambda_2}{2}$$

$$(1-\bar{x})^{-1+i\lambda_4}\bar{x}^{1+i\lambda_2} = \bar{x}^{\bar{h}-\bar{h}_3-\bar{h}_4}\, {}_2F_1\left(\begin{matrix} \bar{h}-\bar{h}_{12}, \bar{h}+\bar{h}_{34} \\ 2\bar{h} \end{matrix}; \bar{x}\right)\Big|_{\bar{h}=1+\frac{i\lambda_2}{2}} .$$

$$G^{21}_{34}(x,\bar{x})_{s,\lambda_1=0} = \sum_{m=1}^{\infty} (a_m f^{a_1a_2b} f^{a_3a_4b} + \tilde{a}_m f^{a_1a_3b} f^{a_2a_4b}) K^{21}_{34}\left[m+\frac{i\lambda_2}{2}, 1+\frac{i\lambda_2}{2}\right]$$

The contributions of t and u channels can be analyzed in a similar way.

$$\Delta = 2 + M + i\lambda_2, \qquad J = M \quad M \geq 0$$

$$K^{21}_{34}(h,\bar{h};x,\bar{x}) = x^{h-h_3-h_4}\, {}_2F_1\left[\begin{matrix} h-h_{12}, h+h_{34} \\ 2h \end{matrix}; x\right] \times \bar{x}^{\bar{h}-\bar{h}_3-\bar{h}_4}{}_2F_1\left[\begin{matrix} \bar{h}-\bar{h}_{12}, \bar{h}+\bar{h}_{34} \\ 2\bar{h} \end{matrix}; \bar{x}\right]$$

Conformal Blocks in CCFT

Lets say we want to discuss crossing symmetry

$$x^{2h_3}\bar{x}^{2\bar{h}_3}G^{21}_{34}(x,\bar{x}) \stackrel{?}{=} G^{31}_{24}(\frac{1}{x},\frac{1}{\bar{x}})$$

But we arrive at an issue:

- correlator before shadow has specific range of cross-ratio **z** depending on space-time channel
- after shadow the new cross-ratio **x** is unrestricted! This is what we would like for CFT crossing but consider i.e. t-channel $(13 \rightleftharpoons 24)_4$ expression:

$$I_t(x) = -\frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{x^2\bar{x}}B(2-i\lambda_2,-i\lambda_4)\,{}_2F_1\left(\begin{matrix}2,2-i\lambda_2\\2+i\lambda_3\end{matrix};\frac{1}{x}\right)$$

$$= -\frac{(x\bar{x})^{\frac{i\lambda_2}{2}}}{\bar{x}}\left[B(-i\lambda_2,-i\lambda_4)\,{}_2F_1\left(\begin{matrix}2,1-i\lambda_3\\1+i\lambda_2\end{matrix};x\right)\right.$$

$$\left.+\,(-x)^{-i\lambda_2}\frac{(1-i\lambda_2)\pi}{\sin(\pi i\lambda_2)}\,{}_2F_1\left(\begin{matrix}2-i\lambda_2,1+i\lambda_4\\1-i\lambda_2\end{matrix};x\right)\right].$$

$$\Delta = 2+M, \qquad J = \Delta - 2 - i\lambda_2, \qquad M \geq 0$$

Imaginary spins! Not single valued correlator, only if restricted "compatible" channels i.e. $x > 1,\ (13 \rightleftharpoons 24)_2 \longrightarrow t-channel\ (13 \rightleftharpoons 24)_4$

Such operators appear in Lorenzian CFT (Light-Ray operators) but we are in Euclidean for the general case we have: $G^{21}_{34}(x,\bar{x})_s \sim f^{a_1a_2b}f^{a_3a_4b}\, I_s + f^{a_1a_3b}f^{a_2a_4b}\, \tilde{I}_s$

$$I_s(x,\bar{x}) = B\left(1+i\lambda_2+i\lambda_4, i\lambda_2+i\lambda_3\right)\ F_1\left[\begin{matrix}1+i\lambda_2+i\lambda_4; 2-i\lambda_1, -i\lambda_1\\ 1-i\lambda_1+i\lambda_2\end{matrix}; x,\bar{x}\right]$$

$$\tilde{I}_s(x,\bar{x}) = B\left(i\lambda_2+i\lambda_4, i\lambda_2+i\lambda_3\right)\ F_1\left[\begin{matrix}i\lambda_2+i\lambda_4; 2-i\lambda_1, -i\lambda_1\\ -i\lambda_1+i\lambda_2\end{matrix}; x,\bar{x}\right]$$

$$G^{21}_{34}(x,\bar{x})_s = \sum_{m,n=0}^{\infty}\left(a_{mn}\, f^{a_1a_2b}f^{a_3a_4b} + \tilde{a}_{mn}\, f^{a_1a_3b}f^{a_2a_4b}\right) \times K^{21}_{34}\Big(\underbrace{m+1+\frac{i\lambda_2}{2}-\frac{i\lambda_1}{2}}_{h}, \underbrace{n+1+\frac{i\lambda_2}{2}-\frac{i\lambda_1}{2}}_{\bar{h}}\Big)$$

$\Delta = 2 + M + i(\lambda_2 - \lambda_1)$ $M \geq 0$

$J = -M, -M+2, \ldots, M-2, M$

Fan, Fotopoulos, St.St., Taylor, Zhu (2021)

infinite tower of primary fields $\Leftrightarrow$ infinite number of symmetries

cf. also Guevara, Himwich, Pate, Strominger (2021)

Further Directions

- understand infinite number of higher spin states and possible further extensions of symmetry algebra
  - *group representation ?*
  - *yield further symmetry constraints on amplitudes !*

- understand symmetries at quantum level
  - *perhaps protected by non-renormalization theorems ?*
  - *Virasoro central charge (-one-loop regulator ?)*

- high-energy (large λ) limit: string world-sheet = celestial sphere

  celestial $CFT_2 \simeq$ string (free) world-sheet CFT_2

- understanding the nature of 2D CFT on celestial sphere would enable a holographic description of flat spacetime

- uplift AdS_3/CFT_2 holography to $\mathscr{M}_4$

  towards flat space-time holography

THANK YOU!

EXTRAS